Descent Relations and Oscillator Level Truncation Method

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Abstract

We reexamine the oscillator level truncation method in the bosonic String Field Theory (SFT) by calculation the descent relation $\langle V_3|V_1\rangle = \tilde{Z}_3\langle V_2|$. For the ghost sector we use the fermionic vertices in the standard oscillator basis. We propose two new schemes for calculations. In the first one we assume that the insertion satisfies the overlap equation for the vertices and in the second one we use the direct calculations. In both schemes we get the correct structures of the exponent and pre-exponent of the vertex $\langle V_2|$, but we find out different normalization factors $\tilde{Z}_3$.

1 Introduction

The Witten cubic bosonic string field theory (CSFT) [1] exhibits very nice algebraic properties. Fields in this theory belong to the infinite-dimension non-commutative $\mathbb{Z}_2$-graded associative algebra. The action has the Chern-Simons form

$$S = \frac{1}{2}\int \Phi \star Q\Phi + \frac{g}{3}\int \Phi \star \Phi \star \Phi. \tag{1.1}$$

Representations of Witten's algebra in the oscillator and conformal languages were proposed [2, 3] (for a review see [4, 5, 6]).

To formulate Witten's theory in the oscillator representation it is enough to construct only two vertices: $\langle|V_3|\rangle$ which represents $\star$, and $\langle V_1|$ which represents $\int$ [2]. In this representation string fields Φ are realised as ket-vectors $|\Phi\rangle$ in the Fock space. The vertices V_1, V_3 are defined up to two normalization factors that can be absorbed in to the charge and string field redefinitions.

The main difficulty in actual calculations with vertices V_i is that they are defined by infinite-dimensional matrices [2, 7, 8, 9]. To perform real calculation one can use finite dimensional approximation of these matrices [10]. One calls this method as the oscillator level truncation method.

An alternative method is the field level truncation method which originates from [11, 12]. This method was used with great success in refs. [13, 14, 15] for tachyon potential calculations. The quartic term in the tachyon potential can be also obtained from the Born zero momentum 4-point CSFT-diagram [10, 11]. It can be also computed within the oscillator level truncation method and the result is in a good agreement with the tachyon potential obtained within field level truncation method.

To justify more the oscillator level truncation method it is worth to check within this method some exact relations. In [16] it was proposed to check descent relations between string vertices. Descent relations [2] relate N-string vertices, $N \geq 1$ in a natural way:

$$\langle V_{N+1}|V_1\rangle = \langle V_N|. \tag{1.2}$$

In ref. [16, 17] it was found that these relations has the form $\langle V_3|V_1\rangle = \tilde{Z}_3\langle V_2|$, where $\tilde{Z}_3 \neq 1$. The appearance $\tilde{Z}_N \neq 1$ in descent relations is considered as an anomaly. We'll discuss this interpretation in section 2. In order to understand the origin of the $\tilde{Z}_N \neq 1$ breaking one has to investigate the vertex descent relations in more details. It is true that one cannot a priori expect $\tilde{Z}_N = 1$ for arbitrary vertices V_1, V_N and V_{N+1}. We present a self-consistence scheme to define vertices $\langle\hat{V}_N|$, $N \geq 2$ in terms of $\langle V_1|$ and $\langle|V_3|\rangle$ which obey (1.2). For vertices defined via overlap equations one can not expect (1.2) since solutions of overlaps are defined up to a number. Generally, instead of (1.2) one has the descent relation of the form

$$\langle V_{N+1}|V_1\rangle = Z_N Z_{-1}^{-1} Z_{N+1}^{-1}\langle V_N|. \tag{1.3}$$

This factorization was found in [18] as a result of direct calculations with vertices in continuous κ-basis [19]. Note that it is more convenient to care out some calculations analytically when the matrices are presented in the diagonal form in the continuous κ-basis [17].

We verify the descent relation for $\langle V_2|V_1\rangle$ and $\langle V_3|V_1\rangle$ in Section 3. In the first case the calculation is similar to that of [2] and gives $\tilde{Z}_2 = 1$ without any truncation. In the second case we use two schemes in the ghost sector, one (labeled by (1)) with an overlap equation for the mid-point insertion and the second (labeled by (2)) without this trick. Both our schemes differ from [16] in the way we treat the ghost zero modes. These schemes give the descent relation in the form

$$\langle V_3^m|V_1^m\rangle = \tilde{Z}_3^m(n)\langle V_2^m|, \quad \langle V_3^{gh}|V_1^{gh}\rangle^{(i)} = \tilde{Z}_{3(i)}^{gh}(n)\langle V_2^{gh}|, \quad i = 1,2 \tag{1.4}$$

where $\tilde{Z}_3^m(n)$ and $\tilde{Z}_3^{gh}(n)$ are level n dependent constants. For $n \to \infty$ the matter part $\tilde{Z}_3^m(n) \to 0$, but the ghost part $\tilde{Z}_3^{gh}(n) \to \infty$. Combining the matter sector with the ghost sector we get the normalization $\tilde{Z}_3$ as $\tilde{Z}_3(n) = \tilde{Z}_3^{gh}(n)\tilde{Z}_3^m(n)$. We numerically prove that $\tilde{Z}_3(n)$ has a finite limit as $n \to \infty$ in both our schemes. The surprising fact is that in two different schemes of calculations we get two different limits, namely $\tilde{Z}_{3(1)}(\infty) \cong 0.070075$ in the scheme with overlap and $\tilde{Z}_{3(2)}(\infty) \cong 0.199381$ in the scheme without overlap. The second result $\tilde{Z}_{3(2)}(\infty)$ coincides with the result obtained in [16].

2 *Set Up*

In the operator representation string fields Φ are realized as ket vectors $|\Phi\rangle$ in the Fock space $\mathcal{H}$ of matter and ghost modes. For the discussion below it is convenient to use a graphic representation similar to a standard diagram technique used to calculate vacuum expectation values of products of Wick monomials. In this technique a Wick monomial is represented by an arrow, so that creation operators correspond to the arrows going to the left and annihilation operators — to the arrows going to the right. Here for simplicity we use one arrow for all creation or annihilation operators of a string. So we represent bra vectors by a point with the right arrows and ket vectors by a point with the left arrows, i.e. we represent a state $|\Phi\rangle$ as

$$|\Phi\rangle = \longleftarrow\!\!\bullet$$

To formulate Witten's theory [1] it is necessary and sufficient to introduce only two vertices:

$${}_{12}\langle|V_3|\rangle_3 \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$$

which represents $\star$, and

$$\langle V_1| \in \mathcal{H}^*$$

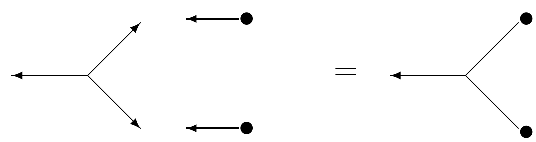
which represents $\int$.

In the graphic notations $\langle V_1|$ is represented by an arrow outgoing from a circle to the right

$$\langle V_1| = $$

and $\langle V_1|\Phi\rangle$ is represented as

The Witten vertex ${}_{12}\langle|V_3|\rangle_3$ is represented by three arrows outgoing from a common point. Two of them are going to the right and one is going to the left. So we can display the star product as follows:

$|\Phi_2\rangle \star |\Phi_1\rangle \; =$ 

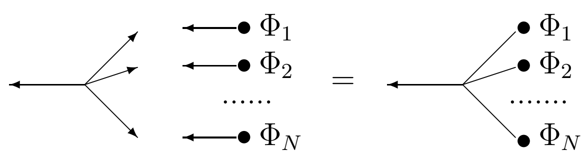
$$= \quad {}_{12}\langle|V_3|\rangle_3|\Phi_2\rangle|\Phi_1\rangle \quad = \quad |\Phi_2 \star \Phi_1\rangle_3.$$

One can build an infinite tower of vertices ${}_{1..N}\langle|V_{N+1}|\rangle_{N+1}$ by gluing of $N-1$ vertices V_3. This tower graphically has the form of a tree with $N+1$ arrows. Inside this tree we can glue free ends of the vertices V_3 arbitrary, all of these gluing are equivalent (it follows from the associativity of the star product [2]) and we will use the simplest case.

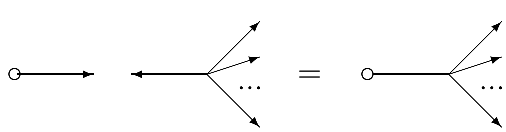
Using the vertex ${}_{1..N}\langle|V_{N+1}|\rangle_{N+1}$, one can define the star product for N fields as:

$$ {}_{1..N}\langle|V_{N+1}|\rangle_{N+1}|\Phi_N\rangle ... |\Phi_1\rangle \equiv |\Phi_N\rangle \star ... \star |\Phi_1\rangle. \tag{2.5}$$

The LHS of (2.5) can be displayed as



One can also build an infinite tower of vertices ${}_{1...N}\langle V_N|$ associated with constructed above vertices ${}_{1..N}\langle|V_{N+1}|\rangle_{N+1}$ by adding one more $\langle V_1|$ as shown below,



Let us call the set of these vertices as Witten's tower of vertices and denote them by a hat. For consistency we will also put the hat on the initial vertices $\langle V_1|$ and $\langle|V_3|\rangle$.

Witten's tower is defined by $\langle \hat{V}_1|$ and $\langle|\hat{V}_3|\rangle$ which are defined up to a number. Therefore one needs two constraints to fix Witten's tower unambiguously. The most natural way is to postulate SFT-CFT [3] correspondence for two vertices:

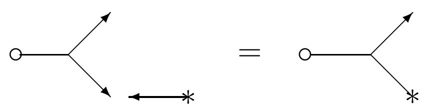
$$\langle \hat{V}_N|\Phi_1\rangle ... |\Phi_N\rangle \equiv \langle \Phi_1 ... \Phi_N\rangle_{\Sigma_N}, \quad N = 2, 3. \tag{2.6}$$

where Σ_N is an appropriate Riemann surface [3]. It is unclear does this fixing of ambiguities also guarantee that (2.6) holds for all vertices ($N > 3$) in the tower.

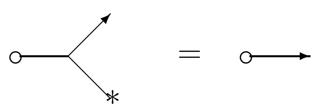
We find also "ket" $|\hat{V}_1\rangle$ as a solution of the "descent relation" [2]

$$\langle \hat{V}_2|\hat{V}_1\rangle = \langle \hat{V}_1|. \tag{2.7}$$

This defining equation for $|\hat{V}_1\rangle$ can be represented graphically. We display a vertex corresponding to $|\hat{V}_1\rangle$ by an arrow outgoing from $*$ to the left. Wick theorem gives that the LHS of (2.7) can be presented as



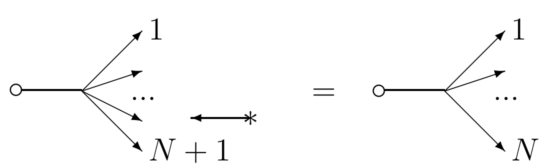
In this notations equation (2.7) looks like



With $|\hat{V}_1\rangle$ subject to (2.7) we are going to verify the descent relations

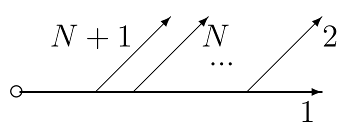
$$\langle \hat{V}_{N+1}|\hat{V}_1\rangle = \langle \hat{V}_N|. \tag{2.8}$$

It can be represented by the graphs for $N > 1$ as



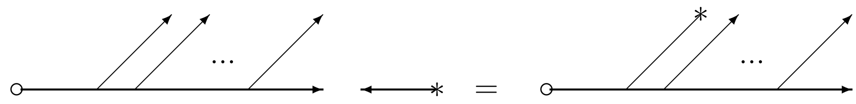
To prove (2.8) we have to use our construction for vertices $\langle \hat{V}_{N+1}|$ (as we have mentioned above, we can take a simplest graph), namely

$\langle \hat{V}_{N+1}| \quad =$

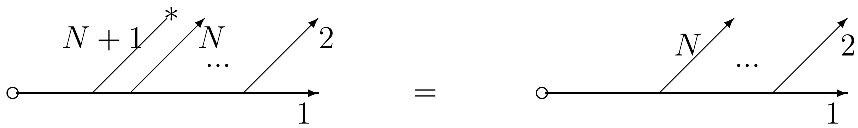


Now

$\langle \hat{V}_{N+1}|\hat{V}_1\rangle$ =



Taking into account (2.7) we get that



that exactly gives $\langle \hat{V}_N|$.

Bra vertices $\langle V_i|$ are usually found as the solutions of the overlap equations [2, 7, 9, 20]. However, these solutions are defined up to a numerical factors, i.e. the vertex $\langle \hat{V}_N|$ in Witten's tower differs from any other vertex $\langle V_N|$ by a number $\langle \hat{V}_N| = Z_N \langle V_N|$ and $|\hat{V}_1\rangle = Z_{-1}|V_1\rangle$. Hence the descent relations for $\langle V_N|$ look like

$$\langle V_{N+1}|V_1\rangle = Z_N Z_{-1}^{-1} Z_{N+1}^{-1} \langle V_N|. \tag{2.9}$$

The factorization of coefficient in descent relation was first found in [18], where vertices were built in the continuous κ - basis. It turned out that descent relation contains the coefficient not equal to one, and moreover this coefficient admits a factorization on three factors. Each of these factors was interpreted as boundary entropy of CFT. Using this factorization in [18] the vertices were redefined as $\langle\langle V_N| = Z_N\langle V_N|$. After this redefinition the descent relation has the coefficient equal to one, $\langle\langle V_{N+1}|V_1\rangle\rangle = \langle\langle V_N|$.

3 Descent Relations for $N = 1$ and $N = 2$

3.1 $\langle V_2|V_1\rangle$

The vertices which will be used were built in [2, 7, 8, 20, 6]. Here and below we use the vectors notations for the modes of the matter fields $X^\mu(\sigma)$ as[1] $a = (a_0, a_1, ...)$, $a^* = (a_0^*, a_{-1}, ...)$ and for the ghost fields $b(\sigma)$, $c(\sigma)$ as b, b^* and c, c^* similarly (see Appendix). In some cases we will include the zero mode b_0 into b with a special comment. The zero mode c_0 enter neither c nor c^*. Also we use the matrix $S_{nk} = (-)^{n+1}\delta_{nk}$ and here $n, k \geq 0$.

We start with the matter sector. The explicit oscillator representation of V_1 and V_2 are given by

$$|V_1^m\rangle = \exp\left\{\frac{1}{2}a^{*1}Sa^{*1}\right\}|0\rangle_1,\ \langle V_2^m| = {}_{1,2}\langle 0|\exp\left\{a^1Sa^2\right\},\ \langle V_1^m| = {}_1\langle 0|\exp\left\{\frac{1}{2}a^1Sa^1\right\}.$$

The LHS of (2.9) is

$$\langle V_2^m|V_1^m\rangle = {}_{1,2}\langle 0|e^{a^1Sa^2}e^{\frac{1}{2}a^{1*}Sa^{1*}}|0\rangle_1 = {}_2\langle 0|e^{\frac{1}{2}a^2SSSa^2} = {}_2\langle 0|\exp\left\{\frac{1}{2}a^2Sa^2\right\} = \langle V_1^m|.$$

[1]We suppress the space-time index below.

Here the Wick theorem [21] is used.

Now we consider the descent relation in the ghost sector. For this calculations we take the following vertices (in this case $n,m\geq 1$):

$$|V_1^{gh}\rangle = \frac{i}{4}b_+^1 b_-^1 \exp\left\{b^{*1}Sc^{*1}\right\}|+\rangle_1, \quad \langle V_1^{gh}| = {}_1\langle +|\exp\{-b^1 Sc^1\}b_-^1 b_+^1\frac{i}{4},$$
$$\langle V_2^{gh}| = {}_{1,2}\langle +|(b_0^1 - b_0^2)\exp\left\{-b^1Sc^2 - b^2Sc^1\right\}.$$

The mid-point insertions are

$$b_+ \equiv b(\pi/2) = \sum_{-\infty}^{\infty} i^n b_n, \qquad b_- \equiv b(-\pi/2) = \sum_{-\infty}^{\infty} i^{-n} b_n.$$

Let us write down the LHS of the descent relation

$$\langle V_2^{gh}|V_1^{gh}\rangle = {}_{1,2}\langle +|(b_0^1 - b_0^2)e^{-b^1Sc^2 - b^2Sc^1}\frac{i}{4}b_+^1 b_-^1 e^{b^{1*}Sc^{1*}}|+\rangle_1.$$

We use the overlap equation for b_+ and b_- to change the index of insertions from “1” to “2”. To exchange the order of the vacuum states we have to take into account their parity, i.e.

$${}_{1,2}\langle +| = -{}_{2,1}\langle +|.$$

So using [21] we have

$$\langle V_2^{gh}|V_1^{gh}\rangle = -{}_2\langle +|e^{-b^2SSSc^2}b_+^2 b_-^2\frac{i}{4} = -{}_2\langle +|\exp\{-b^2Sc^2\}b_+^2 b_-^2\frac{i}{4} = \langle V_1^{gh}|. \tag{3.10}$$

Hence for the vertices which include the matter and ghost sectores the descent relation with coefficient $\tilde{Z}_2$ equals to one has the form

$$\langle V_2|V_1\rangle = \langle V_1|. \tag{3.11}$$

We can conclude that the vertices $\langle V_1|$, $\langle V_2|$ and $|V_1\rangle$ can be considered as the vertices from some Witten's tower.

3.2 $\langle V_3|V_1\rangle$

In this subsection we will consider the descent relation for vertices $\langle V_3|$ and $|V_1\rangle$. This calculation is not so simple as above.

We perform our calculation of the descent relation starting with the ghost sector (notations are listed in Appendix)

$$\langle V_3^{gh}|V_1^{gh}\rangle = N_3\frac{i}{4}{}_{321}\langle +|e^{-b^r X^{rs}c^s}b_+^1 b_-^1 e^{b^{*1}Sc^{*1}}|+\rangle_1. \tag{3.12}$$

Here we include the zero mode b_0 into b. In calculations we drop the coefficient $\frac{i}{4}N_3 = \frac{i}{4}\frac{3\sqrt{3}}{4}$, it will be restored in the final expression.

We will explore two schemes to evaluate (3.12). We can take the insertion $b^1_+ b^1_-$ as it stands or we can use the overlap equation for $\langle V^{gh}_3|$

$$\langle V^{gh}_3|(b^{i-1}(\sigma) - b^i(\pi-\sigma)) = 0, \qquad i = 1, 2, 3 \tag{3.13}$$

to move b^1_+ and b^1_- to the Fock spaces with index "2" or "3". The first scheme of calculations will be presented in Section 3.2.2 and the second one in the Section 3.2.1.

In [16] there was presented another scheme where the vertex $|V_1\rangle$ was considered in $|-\rangle$ vacuum and the direct calculations were performed.

3.2.1 "Overlap Calculations" in Ghost Sector

We use the overlap equation to change the index of insertion b_+ from "1" to "2" and b_- from "1" to "3". So the expression we have to evaluate reads (here b^r include b^r_0):

$$_{321}\langle +|e^{-b^r X^{rs} c^s} e^{b^{*1} S c^{*1}}|+\rangle_1 b^2_+ b^3_-. \tag{3.14}$$

Separating the zero mode b^1_0 we get

$$_{32}\langle +|e^{-W_2}\,{}_1\langle +|(1-b_0 w)e^{-W_1} e^{b^{*1}Sc^{*1}}|+\rangle_1 b^2_+ b^3_- = -_{32}\langle +|e^{-W_2}\,{}_1\langle -|e^{-W_1} w e^{b^{*1}Sc^{*1}}|+\rangle_1 b^2_+ b^3_-.$$

Here we put:

$$w = X^{11}_{0m} c^1_m + X^{1p}_{0m} c^p_m, \quad p = 2, 3$$

in W_1 we have collected the dependence on non-zero modes b^1, c^1 and W_2 depends on the modes of the second and on the third strings (see Appendix).

The zero mode dependence will be of a special interest. In our calculations for any matrix A_{nm} we denote $\bar{A}_k \equiv A_{0k}$ and $\hat{A}_{nm} \equiv A_{nm}$ with $n > 0$.

We need to calculate the following expression

$$_1\langle -|e^{-W_1} w e^{b^{*1}Sc^{*1}}|+\rangle_1. \tag{3.15}$$

To evaluate (3.15) we use the generalized formula of eq. (19) in [21] given in [22, 16]. It has a form:

$$\begin{aligned}\langle -|\exp(-bXc + b\lambda^c + \lambda^b c)\exp(b^* S c^* + b^*\mu^c + \mu^b c^*)|+\rangle = \\ = \det(1-SX)\exp\left\{\mu^b \frac{1}{(1-XS)}(X\mu^c - \lambda^c) + \lambda^b \frac{1}{1-SX}(\mu^c - S\lambda^c)\right\}.\end{aligned} \tag{3.16}$$

Here λ^b, λ^c and μ^b, μ^c are anticommuting vectors.

So using eq. (3.16) and $_{3,2}\langle +| = -_{2,3}\langle +|$ we can evaluate (3.15). Thereby the expression (3.14) gives

$$\langle V^{gh}_3|V^{gh}_1\rangle = -\det(Z^{-1})_{23}\langle +|\bar{U}^{1q} c^q e^{b^p U^{pq} c^q} b^2_+ b^3_-, \tag{3.17}$$

where

$$U^{pq}_{nm} = -X^{p1}_{nk}(ZS)_{kl} X^{1q}_{lm} - X^{pq}_{nm}, \quad \text{here} \quad n \geq 0, \quad m, k, l \geq 1, \quad p, q = 2, 3. \tag{3.18}$$

Here we defined

$$Z = (1 - SX)^{-1},$$

where $X_{nm} \equiv X^{11}_{nm}$, and we stress that n, $m \geq 1$.

To put the expression (3.17) to a normal form we move the insertion $b^2_+ b^3_-$ to the left to the vacua through the exponent and pre-exponential factor. We get the following pre-exponential factor (for some notations see Appendix):

$$\begin{aligned}(b^2K + b_0^2 + b_0^p\bar{U}^{p2}L + b^p\hat{U}^{p2}L)(b^3L + b_0^3 + b_0^p\bar{U}^{p3}K + b^p\hat{U}^{p3}K)\bar{U}^{1q}c^q - \\ -(b^2K + b_0^2 + b_0^p\bar{U}^{p2}L + b^p\hat{U}^{p2}L)\bar{U}^{13}K + (b^3L + b_0^3 + b_0^p\bar{U}^{p3}K + b^p\hat{U}^{p3}K)\bar{U}^{12}L.\end{aligned} \tag{3.19}$$

Because the vacua stays on the left the items with creation operators are dropped.

The matrices $\hat{U}^{pq}$ and rows $\bar{U}^{pq}$ for different p-s and q-s are not independent. By the definition (3.18) and properties of the Neumann matrix (see Appendix) we have:

$$\hat{U}^{i2} + \hat{U}^{i3} = -S, \qquad \hat{U}^{2i} + \hat{U}^{3i} = -S, \qquad i = 2, 3, \tag{3.20}$$

$$\bar{U}^{1j} + \bar{U}^{2j} + \bar{U}^{3j} = 0, \qquad \bar{U}^{j2} + \bar{U}^{j3} = 0, \qquad j = 1, 2, 3. \tag{3.21}$$

For example one can prove the first property [16]

$$\begin{aligned}\hat{U}^{i2} + \hat{U}^{i3} &= -X^{i2} - X^{i1}(1 - SX^{11})^{-1}SX^{12} - X^{i3} - X^{i1}(1 - SX^{11})^{-1}SX^{13} \\ &= -X^{i2} - X^{i3} - X^{i1}(1 - SX^{11})^{-1}S(X^{12} + X^{13}) \\ &= -X^{i2} - X^{i3} - X^{i1}(1 - SX^{11})^{-1}S(S - X^{11}) \\ &= -X^{i2} - X^{i3} - X^{i1}(1 - SX^{11})^{-1}(1 - SX^{11}) = -S.\end{aligned}$$

By using (3.20) an expression (3.19) can be simplified. We express all the matrices in terms of the matrix $\hat{U}^{22}$.

It is useful to introduce the following notations

$$\begin{aligned}\alpha_2 &= (b^2 + b^3)K + (b^2 - b^3)\hat{U}^{22}L, \qquad & \beta_2 &= b_0^2(1 + \bar{U}^{22}L) + b_0^3\bar{U}^{32}L, \\ \alpha_3 &= (b^2 + b^3)L - (b^2 - b^3)\hat{U}^{22}K, \qquad & \beta_3 &= b_0^3(1 + \bar{U}^{33}K) + b_0^2\bar{U}^{23}K.\end{aligned} \tag{3.22}$$

Thereby, the result of moving the b_+b_- insertion to the left (pre-exponential factor (3.19)) can be presented in the following form

$$(\alpha_2 + \beta_2)(\alpha_3 + \beta_3)\bar{U}^{1q}c^q - (\alpha_2 + \beta_2)\bar{U}^{13}K + (\alpha_3 + \beta_3)\bar{U}^{12}L. \tag{3.23}$$

So we have the following result

$$-\det(Z^{-1})_{23}\langle +|[(\alpha_2 + \beta_2)(\alpha_3 + \beta_3)\bar{U}^{1q}c^q - (\alpha_2 + \beta_2)\bar{U}^{13}K + (\alpha_3 + \beta_3)\bar{U}^{12}L]e^{b^pU^{pq}c^q} \tag{3.24}$$

Prima facie it does not match the vertex $\langle V_2^{gh}|$, because this expression does not have the correct structure of vacua and exponent, but it has big pre-exponent and has not correct structure in exponent. Further we'll show the way to get the desired expression.

Below the following relations which are verified numerically (they hold true form the first level) will be useful

$$\bar{U}^{23} = -\bar{U}^{32} S, \quad \bar{U}^{22} = -\bar{U}^{33} S, \quad \bar{U}^{12} = -\bar{U}^{13} S. \tag{3.25}$$

As a result we have:

$$\bar{U}^{32} L = \bar{U}^{23} K, \quad \bar{U}^{22} L = \bar{U}^{33} K, \quad \bar{U}^{13} K = \bar{U}^{12} L, \quad \bar{U}^{32} L = -\bar{U}^{22} K. \tag{3.26}$$

In (3.23) we can find the column $\hat{U}^{22}(L+K)$. For further calculations we need to investigate this object numerically.

Substituting the explicit form of L and K we get the elements of the column as:

$$a_m = 2\sum_{k=0}^{\infty}(-)^k \hat{U}^{22}_{m,2k}.$$

For any level n one can calculate partial sums $a_m(n) = 2\sum_{k=0}^{n}(-)^k \hat{U}^{22}_{m,2k}$ with $m \leq n$. For any fixed m we get a sequence of partial sums $a_m(n)$ as we vary n. We want to prove that $a_m(n) \to 0$ as $n \to \infty$. We prove it for $m = 2, 10, 20, 30, ..., 100$ taking the levels $100 \leq n \leq 300$ with a step equal to 10. The results are plotted at the Figure. We also find fits for the partial sums as $|a_m(n)| \sim \frac{p_m}{n^{q_m}}$ (see Table). Due to the behavior of the fits we can conclude that $|a_m(n)| \to 0$ for $n \to \infty$ and, as a consequence, $a_m \to 0$.

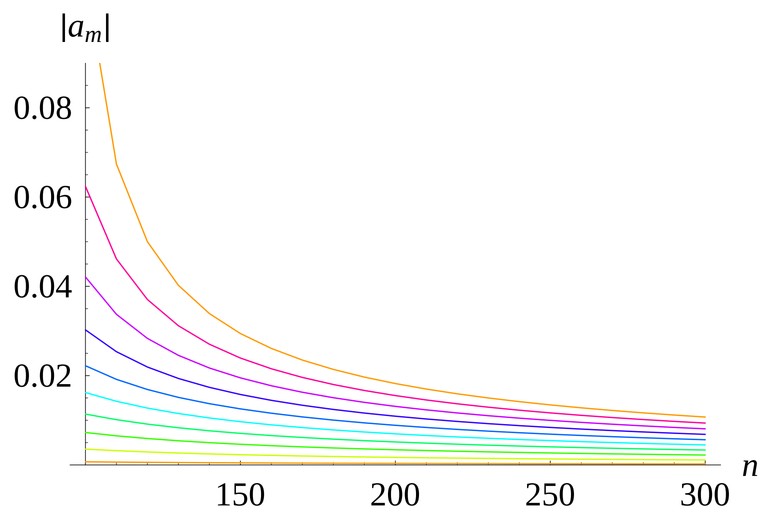


Figure 1:

Due to the information from the Table 1 and Figure 1 we can see that the speed of falling of the elements a_m are increased with extending of m. So we have that

$$\hat{U}^{22}(L+K) = 0. \tag{3.27}$$

Table 1:

m	2	10	20	30	40	50	60	70	80	90	100
p	0.089	0.4617	1.0258	1.844	3.2117	5.806	11.463	26.227	76.781	364.31	16461
q	1.053	1.059	1.077	1.109	1.156	1.221	1.309	1.428	1.595	1.854	2.529

Let us return to (3.23). Substituting α and β into (3.23) we get the expression for the terms linear in b_0. Taking (3.26) and (3.27) into account we get

$$(b_0^2\left[J_0+(b^2\bar{A}_2+b^3\bar{A}_3)\bar{U}^{1q}c^q\right]-b_0^3\left[J_0+(b^2\bar{B}_2+b^3\bar{B}_3)\bar{U}^{1q}c^q\right]), \qquad (3.28)$$

here (we express all $\bar{U}^{12}$ and $\bar{U}^{22}$ via eq. (3.26))

$$\begin{aligned} J_0 &= -(1+\bar{U}^{22}(K+L))\bar{U}^{12}L, \\ \bar{A}^i &= (-)^i\hat{U}^{22}L(1+\bar{U}^{22}(K+L))+L(1+\bar{U}^{22}L)+K\bar{U}^{22}K, \\ \bar{B}^i &= (-)^i\hat{U}^{22}L(1+\bar{U}^{22}(K+L))+K(1+\bar{U}^{22}L)+L\bar{U}^{22}K. \end{aligned} \qquad (3.29)$$

So we see that J_0 contributes to $\tilde{Z}_3^{gh}(n)$. The remaining terms which are quadratic in b_0 are presented in the form:

$$\beta_2\beta_3\bar{U}^{1q}c^q = b_0^2 b_0^3\left[(1+\bar{U}^{22}L)^2-(\bar{U}^{22}K)^2\right]\bar{U}^{1q}c^q. \qquad (3.30)$$

The free part in (3.23) contains of two terms: $\alpha_2\alpha_3\bar{U}^{1q}c^q$ and $(\alpha_3-\alpha_2)\bar{U}^{12}L$. we have to extract the normalization factor J_0 After this rescaling these quantities vanish. On the figures we see the behavior of these two normalized terms for $n\to\infty$.

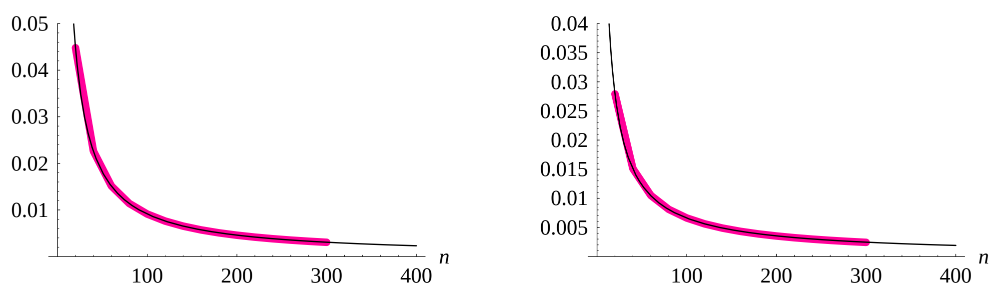


Figure 2:

We find that for n=20...300 this curves are described by the fits $\sim\frac{0.87}{n^{0.99}}$ and $\sim\frac{0.41}{n^{0.9}}$ respectively. So these terms vanish for $n\to\infty$.

Than we are going to rise up the terms linear and quadratic in b_0 term to the exponent. In this way we want to get the necessary structure in the exponent.

$${}_{23}\langle +|J_0(...)e^{bUc}\Longrightarrow {}_{23}\langle +|J_0(b_0^2-b_0^3)e^{bVc}e^{bUc}, \qquad (3.31)$$

(here the zero modes are contained in the matrix V as well as in the matrix U). To get the desired structure the following conditions has to take place (for $n \to \infty$)

$$\bar{V}^{pq} + \bar{U}^{pq} \to 0, \tag{3.32}$$

$$\hat{V}^{pp} + \hat{U}^{pp} \to 0, \quad \hat{V}^{p,p+1} + \hat{U}^{p,p+1} \to -S. \tag{3.33}$$

The constraint (3.32) cancels the total zero modes dependence in the exponent and the constraint (3.33) will give the required structure of quadratic form. Below we'll discuss the realization of this program.

For convenience we'll go backwards, i.e. we will expand $\exp\{bVc\}$ and compare the obtained structure with given by (3.23). In this way we will get conditions to rise up the pre-exponential factor to the exponent. In our argumentations it is important that we suppose the desired matrix $\hat{V}$ to be a tensor product of two vectors linear in ghost modes, i.e. $\hat{V} = \bar{V}_1 \otimes \bar{V}_2$. Due to this guess the exponent e^{bVc} expands up to linear term (the quadratic term equals to zero due to their Grassman structure) only.

Hence we get the following system of equations for unknown matrices V^{pq}, $p,q = 2,3$:

$$1. \quad J_0(\bar{V}^{2q} + \bar{V}^{3q}) = \left[(1 + \bar{U}^{22}L)^2 - (\bar{U}^{22}K)^2\right] \bar{U}^{1q}, \tag{3.34a}$$

$$2. \quad (\bar{V}^{3q} + \bar{V}^{2q})c^q \cdot b^p \hat{V}^{pq} c^q = 0, \tag{3.34b}$$

$$3. \quad \hat{V}^{pq} = \frac{\bar{A}^p}{J_0} \otimes \bar{U}^{1q}, \tag{3.34c}$$

$$4. \quad \hat{V}^{pq} = \frac{\bar{B}^p}{J_0} \otimes \bar{U}^{1q}. \tag{3.34d}$$

This equation do not define the matrices V^{pq} completely but it's enough to rise up the pre-exponential factor to the exponent.

We can mark out two consistency conditions of the system:

$$\frac{\bar{A}^p}{J_0} - \frac{\bar{B}^p}{J_0} = T\frac{(1 + \bar{U}^{22}T)}{J_0} \to 0, \quad p = 2,3. \tag{3.35}$$

We calculated the LHS of (3.35)for $n = 20...300$ and got the fits for this set of points as $\frac{4.3}{n^{0.94}}$ (see Figure 3).

The equation (3.34c) defines $\hat{V}^{pq}$ and equation (3.34a) defines the sum $\bar{V}^{2q} + \bar{V}^{3q}$. The constraints (3.34b) follow from (3.34a) and (3.34c):

$$\frac{1}{J_0}\left[(1 + \bar{U}^{22}L)^2 - (\bar{U}^{22}K)^2\right] \bar{U}^{1q}c^q \cdot b^2 \frac{\bar{A}^p}{J_0} \otimes \bar{U}^{1q}c^q = 0.$$

The equation (3.34a) does not define $\bar{V}^{pq}$-s separately but only the sum. Let us put $\bar{V}^{pq} \equiv -\bar{U}^{pq}$ thus eq. (3.32) is fulfilled. Hence, we have to check the constraint (3.34a) for this choice:

$$-J_0(\bar{U}^{2q} + \bar{U}^{3q}) - [(1 + \bar{U}^{22}L)^2 - (\bar{U}^{22}K)^2]\bar{U}^{1q} = 0. \tag{3.36}$$

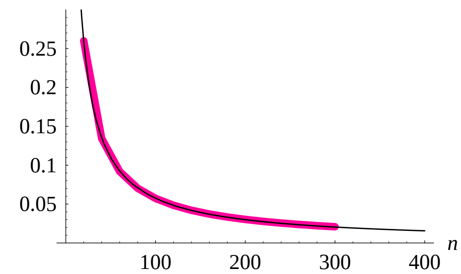


Figure 3:

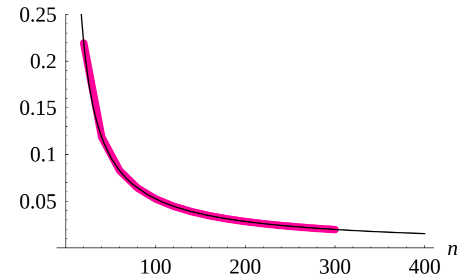


Figure 4:

Due to (3.21) we cancel the factor $\bar{U}^{1q}$ in both sides of eq. (3.36) and have to check numerically the equality

$$J_0 - [(1+\bar{U}^{22}L)^2 - (\bar{U}^{22}K)^2] = 0. \tag{3.37}$$

We checked it up to level 300 and got the fit as $\frac{3.16}{n^{0.89}}$ (see Figure 4).

The eq. (3.34c) defines the matrices $\hat{V}^{pq} = J_0^{-1}\bar{A}^p \otimes \bar{U}^{1q}$. So we have to check the conditions (3.33) for this choice. We do it up to level $n = 300$. For $p = 2$ for the first condition in (3.33) we have the fit $\sim \frac{0.86}{n^{0.82}}$ (for the second condition in (3.33) and for $p = 3$ the calculations and the fits are almost the same). The result for numerical calculations and the fit can be seen on the Figure 5.

Thereby, pre-exponential factor (3.23) rises up to the exponent. These complete the proof of desired structure of the exponential.

Hence we got that the descent relation in the ghost sector has the following form

$$\begin{aligned}
{}_{321}\langle V_3^{gh}|V_1^{gh}\rangle_1 &= -J_0 N_3 \frac{i}{4}\det(1-SX)_{23}\langle +|(b_0^2 - b_0^3)e^{-b^2Sc^3 - b^3Sc^2} \\
&= -J_0 N_3 \frac{i}{4}\det(1-SX)_{23}\langle V_2| \equiv \tilde{Z}^{gh}_{ov}{}_{23}\langle V_2^{gh}|.
\end{aligned} \tag{3.38}$$

Here we restore the coefficients which were dropped above.

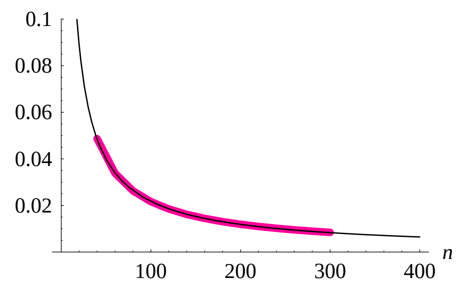


Figure 5:

3.2.2 "Non-Overlap Calculations" in Ghost Sector

Here we examine the descent relation by straightforward calculations without using overlaps.

The starting expression is eq. (3.12)

$$ {}_{321}\langle +|e^{-b^r X^{rs} c^s} b_+^1 b_-^1 e^{b^* S c^*}|+\rangle_1. \tag{3.39}$$

It's useful to rewrite the insertion as

$$ b_+^1 b_-^1 \equiv A + b_0^1 B, \tag{3.40}$$

here we denote (see Appendix)

$$\begin{aligned} A &\equiv bK\cdot bL + bK\cdot b^*K - bL\cdot b^*L + b^*L\cdot b^*K, \\ B &\equiv bT - b^*T, \end{aligned}$$

(temporary we omit superscript 1 for the modes of the first string).

We put (3.40) to (3.39) and isolate the exponent with modes of the second and third strings as $\exp(-W_2)$. So we get

$$ {}_{32}\langle +|e^{-W_2}{}_1\langle +|e^{-W_1 - b_0^1 w}(A + b_0 B)e^{b^* S c^*}|+\rangle_1. $$

Here w, W_1 and W_2 have the same form as in the previous section. Let us expand the exponent in b_0^1 mode (here we drop the factor ${}_{32}\langle +|e^{-W_2}$)

$$\begin{aligned} &{}_1\langle +|e^{-W_1}(1 - b_0 w)(A + b_0 B)e^{b^* S c^*}|+\rangle_1 \\ &= {}_1\langle +|e^{-W_1}(A - b_0 w A + b_0 B)e^{b^* S c^*}|+\rangle_1. \end{aligned}$$

The first term does not contribute. In two remaining terms we move b_0 to the left vacuum:

$$ = {}_1\langle -|e^{-W_1} B e^{b^* S c^*}|+\rangle_1 - {}_1\langle -|e^{-W_1} A w e^{b^* S c^*}|+\rangle_1. \tag{3.41}$$

The direct calculation for (3.41) leads to:

$$\begin{aligned} \langle V_3^{gh}|V_1^{gh}\rangle = -2\det(Z^{-1})_{23}\langle +|\exp\left\{-\lambda^b Z S \lambda^c - \sum_{\substack{n=0,\\ m=1}}^{\infty}\sum_{p,q=2}^{3} b_n^p X_{nm}^{pq} c_m^q\right\} \\ \cdot\left[\lambda^b Z T + 2\bar{X} Z L\cdot\lambda^b Z K - 2\bar{X} Z K\cdot\lambda^b Z L - 2\lambda^b Z L\cdot\lambda^b Z K\cdot\bar{U}^{1q} c^q\right]. \end{aligned}$$

In the exponent we got the matrices U^{pq}_{nm} (see eq. (3.18)) as in the previous paragraph. This is not a surprise because the pre-exponential factor does not contribute to the exponent. For the following calculations we use the property $\bar{X}ZL = \bar{X}ZK$ proved numerically (this property is true for any level). So the result is

$$= -2\det(Z^{-1})_{23}\langle +|e^{-\check{b}^p U^{pq} c^q}[\theta \check{b}^r X^{r1} ZT + 2\check{b}^p X^{p1} ZL \cdot \check{b}^r X^{r1} ZK \bar{U}^{1q} c^q], \tag{3.42}$$

where $\theta \equiv (1 - 2\bar{X}ZL)$ and $\check{b}^r \equiv (b^r_0, b^r_1, b^r_2, ...)$.

It is instructive to stress that $\check{b}^p X^{p1}$ does contain the zero modes as well as $\check{b}^p U^{pq}$ in the exponent.

As for the first method we analyze the structure of the pre-exponential factor in $b_0^{2,3}$. The free from b_0 term has the form

$$\theta b^p \hat{X}^{p1} ZT + 2b^p \hat{X}^{p1} ZL \cdot b^r \hat{X}^{r1} ZK \bar{U}^{1q} c^q. \tag{3.43}$$

Due to the normalization on $\mathfrak{J}_0$ (see below) this term vanishes. On the Figure 6 we can see the behavior of two free terms. As we can see the both terms vanish as $\sim \frac{0.856}{n^{0.819}}$ and $\sim \frac{0.073}{n^{0.142}}$

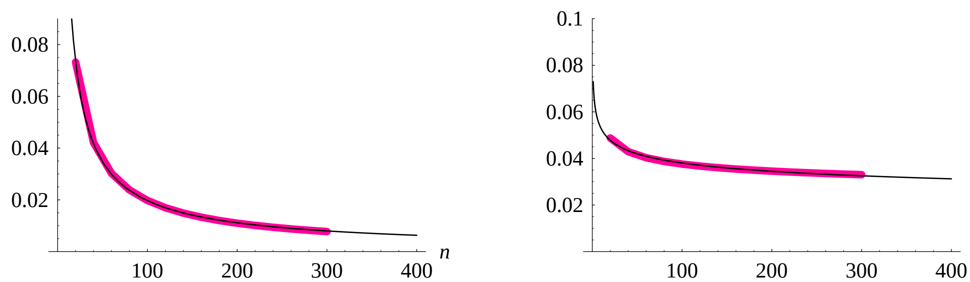


Figure 6:

respectively.

Let us consider the terms proportional to b_0^2 and b_0^3:

$$\begin{aligned}
&-\theta b_0^2 \bar{X}^{21} ZT - \theta b_0^3 \bar{X}^{31} ZT \\
&-2(b_0^2 \bar{X}^{21} ZL \cdot b^p \hat{X}^{p1} ZK + b_0^3 \bar{X}^{31} ZL \cdot b^p \hat{X}^{p1} ZK)\bar{U}^{1q} c^q \\
&+2(b_0^2 \bar{X}^{21} ZK \cdot b^p \hat{X}^{p1} ZL + b_0^3 \bar{X}^{31} ZK \cdot b^p \hat{X}^{p1} ZL)\bar{U}^{1q} c^q \\
&\equiv b_0^2(\mathfrak{J}_0 + b^2 \bar{\mathfrak{A}}^2 \bar{U}^{1q} c^q + b^3 \bar{\mathfrak{A}}^3 \bar{U}^{1q} c^q) + b_0^3(\mathfrak{J}'_0 + b^2 \bar{\mathfrak{B}}^2 \bar{U}^{1q} c^q + b^3 \bar{\mathfrak{B}}^3 \bar{U}^{1q} c^q),
\end{aligned}$$

where we define

$$\begin{aligned}
\mathfrak{J}_0 &= -\theta \bar{X}^{21} ZT, \qquad \mathfrak{J}'_0 = -\theta \bar{X}^{31} ZT, \\
\bar{\mathfrak{A}}^i &= -2(\hat{X}^{i1} ZK \cdot \bar{X}^{21} ZL - \hat{X}^{i1} ZL \cdot \bar{X}^{21} ZK), \quad i = 2, 3, \\
\bar{\mathfrak{B}}^i &= -2(\hat{X}^{i1} ZK \cdot \bar{X}^{31} ZL - \hat{X}^{i1} ZL \cdot \bar{X}^{31} ZK).
\end{aligned} \tag{3.44}$$

For the correct structure of vacua it's necessary but not sufficient to fulfil the condition $\mathfrak{J}'_0 + \mathfrak{J}_0 \to 0$. By numerically calculations: $\mathfrak{J}'_0 + \mathfrak{J}_0 = 0$, starting from the first level.

In the pre-exponential factor we have the term which is quadratic in b_0. It has the form

$$2b_0^2 b_0^3(-\bar{X}^{21}ZL \cdot \bar{X}^{31}ZK + \bar{X}^{31}ZL \cdot \bar{X}^{21}ZK)\bar{U}^{1q}c^q. \tag{3.45}$$

The next step is to rise up the linear and quadratic terms to the exponent as we did in the previous section. In terms of that method we again get some new matrices $V^{pq},\ p = 2, 3$. So we have the conditions (3.32) and (3.33) and the following system of equations:

$$1.\quad \mathfrak{J}_0(\bar{V}^{2q} + \bar{V}^{3q}) = 2(\bar{X}^{21}ZL \cdot \bar{X}^{31}ZK - \bar{X}^{31}ZL \cdot \bar{X}^{21}ZK)\bar{U}^{1q}, \tag{3.46a}$$

$$2.\quad (\bar{V}^{2q} + \bar{V}^{3q})c^q \cdot b^p \hat{V}^{ps}c^s = 0, \tag{3.46b}$$

$$3.\quad \hat{V}^{pq} = \frac{\bar{\mathfrak{A}}^p}{\mathfrak{J}_0} \otimes \bar{U}^{1q}, \tag{3.46c}$$

$$4.\quad \hat{V}^{pq} = \frac{\bar{\mathfrak{B}}^p}{\mathfrak{J}_0} \otimes \bar{U}^{1q}. \tag{3.46d}$$

We get the same structure as in eqs. (3.34). However here we have the equations in terms of the matrices X versus matrices U as in the previous section.

The consistency conditions of the system is

$$\frac{\bar{\mathfrak{A}}^r}{\mathfrak{J}_0} + \frac{\bar{\mathfrak{B}}^r}{\mathfrak{J}_0} \to 0, \quad r = 2, 3.$$

The numerical calculations are summarized in Figure 7. Data produce the fit $\sim \frac{0.705}{n^{0.783}}$, i.e. the data seem to indicate that the sum vanishes as $n \to \infty$.

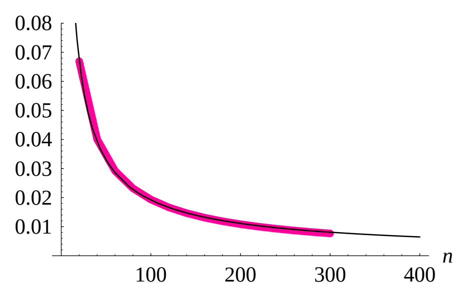


Figure 7:

To solve the system (3.46) we use the same strategy as in the previous section.

Let us stress that the system (3.46) defines only the sum of the rows $\bar{V}^{2q} + \bar{V}^{3q}$. We set $\bar{U}^{pq} = -\bar{V}^{pq}$ thus the equation (3.32) is fulfilled. So the equation (3.46a) in the system transforms into

$$\mathfrak{J}_0(\bar{U}^{2q} + \bar{U}^{3q}) = -2(\bar{X}^{21}ZL \cdot \bar{X}^{31}ZK - \bar{X}^{31}ZL \cdot \bar{X}^{21}ZK)\bar{U}^{1s}.$$

Using $\bar{U}^{1q} = -(\bar{U}^{2q} + \bar{U}^{3q})$, rewrite it as

$$\mathfrak{J}_0 = 2(\bar{X}^{21}ZL \cdot \bar{X}^{31}ZK - \bar{X}^{31}ZL \cdot \bar{X}^{21}ZK)$$

or

$$-(1-2\bar{X}ZL)\bar{X}^{21}Z(L-K)=2(\bar{X}^{21}ZL\cdot\bar{X}^{31}ZK-\bar{X}^{31}ZL\cdot\bar{X}^{21}ZK).$$

On the Figure 8 we present the behavior of this equation rewritten in the form

$$1-2\frac{\bar{X}^{21}ZL\cdot\bar{X}^{31}ZK-\bar{X}^{31}ZL\cdot\bar{X}^{21}ZK}{\mathfrak{J}_0}$$

for $n\to\infty$. For this graph we have the fit $\sim\frac{1.13}{n^{0.86}}$.

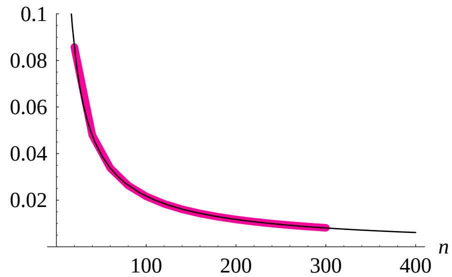


Figure 8:

Eq. (3.46c) in the system defines $\hat{V}^{pq}$ and we can check the conditions (3.33). The numerical calculations shows these conditions to behave similarly. So we give only one result for the first expression $Max|\hat{U}^{22}+\hat{V}^{22}|$ for $n=40,...,300$. We discard the last 20 rows and columns in the matrices in our calculations.

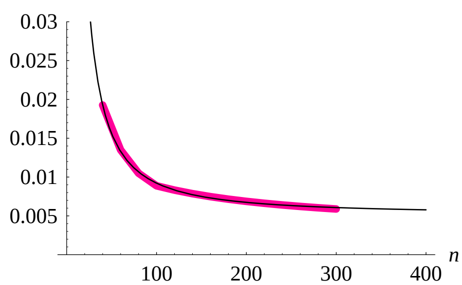


Figure 9:

For $Max|\hat{U}^{22}+\hat{V}^{22}|$ we got the fit of the form $\sim 0.005+\frac{2.17}{n^{1.36}}$.
Hence we got that the descent relation in the ghost sector has the following form

$$\begin{aligned}{}_{321}\langle V_3^{gh}|V_1^{gh}\rangle_1 &= \mathfrak{J}_0 N_3\frac{i}{2}\det(1-SX)_{23}\langle+|(b_0^2-b_0^3)e^{-b^2Sc^3-b^3Sc^2}\\ &= \mathfrak{J}_0 N_3\frac{i}{2}\det(1-SX)_{23}\langle V_2^{gh}|\equiv\tilde{Z}^{gh}_{nov\,23}\langle V_2^{gh}|.\end{aligned}\qquad(3.47)$$

Here we restore the coefficients which were dropped above.

3.2.3 Result with Matter Sector

The normalization $\tilde{Z}_3$ in the descent relation $\langle V_3|V_1\rangle$ consists of two factors: ghost $\tilde{Z}^{gh}$ and matter $\tilde{Z}^m$ ones, i.e. $\tilde{Z}_3 = \tilde{Z}^m\tilde{Z}^{gh}$. In the ghost sector in the "overlap calculations" the normalization is

$$\tilde{Z}^{gh}_{ov} = -\frac{i}{4}N_3 J_0 \det(1-SX), \tag{3.48}$$

in the "non-overlap calculations" we got

$$\tilde{Z}^{gh}_{nov} = \frac{i}{2}N_3 \mathfrak{J}_0 \det(1-SX). \tag{3.49}$$

In the matter sector we have the following normalization factor

$$\tilde{Z}^m = \det(1+V^{11m}S)^{-13}. \tag{3.50}$$

In (3.50) the matrix V^{11m} is the Neumann matrix for matter [2, 10]. This result was given in [16] and we checked its accuracy. Because of the simple structure of the vertices in the matter sector one can directly use [21] without any intermediate calculations.

Thus the numerical calculations of the full normalization factors give us that we have two different results which are displayed on the Figure 10.

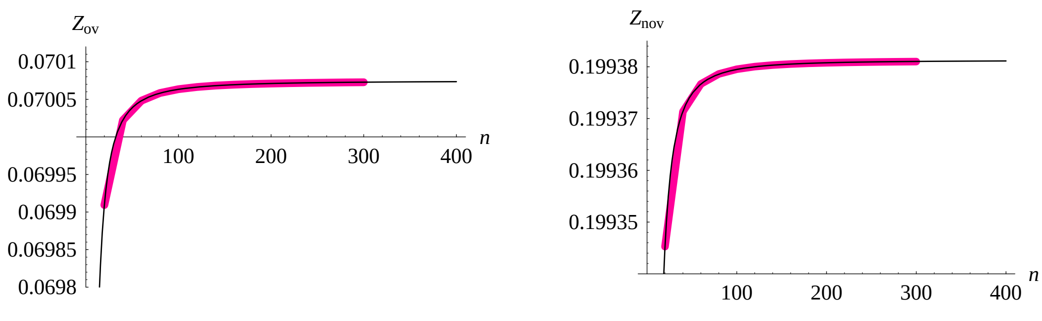


Figure 10:

We have the following fits for these graphs: for the first, "overlap calculations" we have $\tilde{Z}_3 \sim 0.070075 - \frac{0.024}{n^{1.666}}$; for the second, "non-overlap calculations" we have $\tilde{Z}_3 \sim 0.199381 - \frac{0.01}{n^{1.87}}$.

To conclude we calculate the descent relation for $\langle V_3|V_1\rangle$ using two different schemes. Both calculations are performed with using the oscillator level truncation method and differ in a treatment of the mid-point insertion. Both schemes gives the vertex $\langle V_2|$ with two different normalization factors. A similar problem with the oscillator level truncation method was been found in [23].

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Appendix

In this article we use the notations for the vacua following [2]

in the matter sector:

$$a_n|0\rangle = 0, \qquad n \geq 0, \qquad \langle 0|0\rangle = 1,$$

in the ghost sector:

$$\begin{aligned} c_n|+\rangle &= 0, \quad n \geq 0, \\ b_m|+\rangle &= 0, \quad m \geq 1, \end{aligned}$$

$$|-\rangle = b_0|+\rangle, \qquad \langle -|+\rangle = \langle +|-\rangle = 1.$$

The vertex $\langle V_3|$ is given by:

$$\langle V_3| = {}_{321}\langle +| \exp\left\{ -\sum_{r,s=1}^{3} \sum_{\substack{n=0,\\ m=1}} b_n^r X_{nm}^{rs} c_m^s \right\} N_3,$$

where

$${}_{321}\langle +| \equiv {}_3\langle +|_2\langle +|_1\langle +|.$$

Neumann matrix obey the following properties [2, 21]

$$\begin{aligned} X_{nm}^{r,r} + X_{nm}^{r,r+1} + X_{nm}^{r,r-1} &= S_{nm}, \quad n \neq 0 \\ X_{0m}^{r,r} + X_{0m}^{r,r+1} + X_{0m}^{r,r-1} &= 0. \end{aligned}$$

here $X_{nm}^{r,r+2} = X_{nm}^{r,r-1}$ we have the same formula for the second fixed index. We separate the zero mode in the insertion $b_+ b_-$ as

$$b_+ \equiv b(\pi/2) = \sum_{-\infty}^{\infty} i^n b_n = bK + b_0 + b^* L, \quad b_- \equiv b(-\pi/2) = \sum_{-\infty}^{\infty} i^{-n} b_n = bL + b_0 + b^* K,$$

where

$$\begin{aligned} &b^j \equiv (b_1^j, b_2^j, ...), \quad b^{j*} \equiv (b_{-1}^j, b_{-2}^j, ...), \quad c^{j*} \equiv (c_{-1}^j, c_{-2}^j, ...)^T, \quad c^j \equiv (c_1^j, c_2^j, ...)^T, \\ &K \equiv (i, -1, -i, 1, ...)^T, \quad L \equiv (-i, -1, i, 1, ...)^T, \quad T \equiv L - K = (-2i, 0, 2i, 0, ...)^T \end{aligned}$$

In the vertex V_3 we separate the dependence on the first string as follows

$$\begin{aligned} W_1 &= \sum_{n,m=1}^{\infty} b_n^1 X_{nm}^{11} c_m^1 + \sum_{s=2}^{3} \sum_{n,m=1}^{\infty} b_n^1 X_{nm}^{1s} c_m^s + \sum_{r=2}^{3} \sum_{\substack{n=0,\\ m=1}}^{\infty} b_n^r X_{nm}^{r1} c_m^1 \\ &\equiv b^1 \hat{X} c^1 + b^1 \lambda^c + \lambda^b c^1. \end{aligned}$$

where

$$\lambda^c \equiv \sum_{s=2}^{3} \hat{X}^{1s} c^s, \quad \lambda^b \equiv \sum_{r=2}^{3} (b_0^r \bar{X}^{r1} + b^r \hat{X}^{r1}),$$

and

$$W_2 = \sum_{n,m=1}^{\infty} b_n^p X_{nm}^{11} c_m^q, \qquad p, q = 2, 3.$$

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